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S-matrices in integrable models of isotropic magnetic chains: I*

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Abstract. We show the space of n-magnon states in integrable isotropic models of magnetic chains, the structure of the space of states in the RSOS model. The S-matrix in these models is described in terms of weights in integrable RSOS models.

1. Introduction

During the last few years many works were written regarding the studies of integrable Heisenberg magnetics of XXX-type of spin S. This model at $S=\frac{1}{2}$ coincides with the usual Heisenberg magnet [1]. For $S>\frac{1}{2}$, this model related to SU(2) invariant solutions of the Yang–Baxter equation [2]. This relation was based on well known relations between 1D quantum field theory and 2D classical statistical mechanics [3].

One of the most interesting properties of these models is their critical behaviour. The infrared asymptotics of correlators in these models are described by conformal field theory with the central charge $c=3S/S+1$, where S is the spin of the magnet. This value of the central charge correspondss to the SU(2) Wess–Zumino–Witten conformal field theory of the level $k=2S$.

The approach based on the solutions of the Yang–Baxter equations is very effective for the study of quantum integrable systems and it is known as the quantum inverse scattering method [5, 6]. This method is very effective for further generalizations of the Heisenberg models. In this specific case when individual spins transform according to certain representations of any simple Lie algebras [7–9]. Obviously these solutions off the Yang–Baxter equations can be considered as representations of certain Hopf algebras known as Yangians [10, 11].

In many papers devoted to various aspects of $\mathscr{G}$-invariant magnet where $\mathscr{G}$ is a Lie algebra this anomaly was observed in dimensions of states of physical particles. The dimensions of n-magnon space in this model cannot be described as $\mathscr{D}^n$ where $\mathscr{D}$ is the dimension of 1-magnon space. It is also less than any considered subspace which is invariant under the action of any natural symmetry. In [12] quasiclassical arguments were given explaining this anomaly for $S=1$ as a consequence of the fermionic anomaly [13]. Here the last factor is the S-matrix in the chiral Gross–Neve model [16] and the matrix $S^{(\mathrm{RSOS})}$ is defined in terms of Boltzmann weights of the critical RSOS model [17]. These matrices are described in the next section.

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2. SU(2)-invariant magnet

The Hamiltonian of the integrable XXX Heisenberg magnet of the spin S acts in the space $\mathscr{H}_N = (\mathbb{C}^{2S})^{\otimes N}$ and has the following form

$$H = \sum_{n=1}^{N} p_S(S_n, S_{n+1}) \tag{2.1}$$

where $S_n^\alpha = 1 \otimes \ldots \otimes S^\alpha \otimes \ldots \otimes$ is a generator of SU(2) acting in the nth factor of $(\mathbb{C}^{2S+1})^{\otimes N}$, $S_{N+1}^\alpha \equiv S_1^\alpha$. The polynomial $p_S(x)$

$$p_S(x) = 2 \sum_{l=0}^{2S} \sum_{k=l+1}^{2S} \frac{1}{k} \prod_{\substack{j \neq k \\ j=0}}^{2S} \frac{x - x_j}{x_l - x_j} \tag{2.2}$$

has the degree $2S$. Here $x_l = \frac{1}{2}l(l+1) - S(S+1)$.

The Hamiltonian is a logarithmic derivative of the transfer matrix of the vertex model

$$H = \frac{\mathrm{d}}{\mathrm{d}u} \log t(u)|_{u=0} \qquad t(u) = \mathrm{tr}_0(R_{01}(u) \ldots R_{0N}(u)) \tag{2.3}$$

where $R_{0i}(u)$ is a standard notation for the $(2S+1) \times (2S+1)$ matrix $R(u)$ acting non-trivially only in 0th and ith factors of $(\mathbb{C}^{2S+1})^{\otimes(N+1)}$. The matrix $R(u)$ is a SU(2)-invariant solution of the Yang–Baxter equation [2]

$$R(u) = \sum_{l=0}^{2S} \left(\prod_{k=l+1}^{2S} \frac{u-k}{u+k} \right) \prod_{\substack{j \neq l \\ j=0}}^{2S} \frac{\sigma - x_j}{x_l - x_j} \tag{2.4}$$

where x_j is defined above and $\sigma = \Sigma_{\alpha=1}^{3} S^\alpha \otimes S^\alpha \equiv \boldsymbol{S} \otimes \boldsymbol{S}$.

As was shown in [18] the spectrum of the transfer-matrix (2.3) and of the Hamiltonian (2.1) is parametrized by solutions of so-called Bethe equations. This is the system of equations for n complex numbers α_j

$$\left(\frac{\alpha_j - iS}{\alpha_j + iS} \right)^N = \prod_{k \neq j}^{n} \frac{\alpha_j - \alpha_k + i}{\alpha_j - \alpha_k - i}. \tag{2.5}$$

The eigenvalues of H corresponding to the solution $\alpha_1 \ldots \alpha_n$ are†

$$E = -\sum_{j=1}^{n} \frac{2S}{\alpha_j^2 + S^2}. \tag{2.6}$$

The operator

$$t(0) = P_{12} \ldots P_{1N}$$

where P_{ij} is the permutation of ith and jth factors in $(\mathbb{C}^{2S+1})^{\otimes 1}$, gives the translation operator. The momentum of the state is natural to define as eigenvalues of $1/i \log t(0)$. This operator commutes with H and has the eigenvalue

$$K = \sum_{j=1}^{n} 2 \tan^{-1}\left(\frac{\alpha_j}{S} \right)$$

on the state parametrized by numbers $\alpha_1, \ldots, \alpha_n$.

† The eigenvalue corresponding to each solution of (1.5) with fixed n is degenerate with degree $2(NS - n) + 1$ [19]. The corresponding eigenspace is the space of the irreducible representation of SU(2) of the spin $NS - n$.

In the thermodynamical limit $N\to\infty$ solutions of the system (2.5) have so-called string behaviour. This means that for fixed n in this limit each solution $\{\alpha_j\}_{j=1}^n$ has the form

$$\{\alpha_j\}_{j=1}^n = \bigcup_{n=\sum_{k\geq 1} k\nu_k} \left\{\left\{\alpha_j^{(k)}+\frac{i}{2}(k+1-2t)\right\}_{t=1}^{k}\right\}_{j=1}^{\nu_k}.$$

Moreover, almost any solution of the system (2.5) has such a structure if $N\to\infty$, $n/N<S$†. In this limit, numbers $\alpha_j^{(k)}$ (centres of strings) become distributed along the real axis with densities $\rho_k(\lambda)$, such that $\rho_k(\lambda)\,d\lambda$ is the number of $\alpha_j^{(k)}$ in the interval $d\lambda$. According to the classical work by Yang and Yang [25] one can introduce the density of holes $\tilde{\rho}_k(\lambda)$. In the thermodynamic limit the system (2.5) gives the system of integral equations for densities $\rho_k(\lambda)$ and $\tilde{\rho}_k(\lambda)$

$$a_{n,2S}(\lambda)=\tilde{\rho}_n(\lambda)+\sum_{k\geq 0} A_{nk}*\rho_k(\lambda) \tag{2.7}$$

where functions $a_{n,2S}(\lambda)$ and $A_{n,k}(\lambda)$ are defined in the appendix,

$$a*b(\lambda)=\int_{-\infty}^{+\infty} a(\lambda-\mu)b(\mu)\,d\mu.$$

The energy and momentum of the thermodynamic state characterized by densities $\rho_j(\lambda)$ have the form

$$\begin{aligned} E(\rho)&=-N\int_{-\infty}^{+\infty}\sum_{j\geq 1} a_{j,2S}(\alpha)\rho_j(\alpha)\,d\alpha \\ K(\rho)&=N\int_{-\infty}^{+\infty}\sum_{j\geq 1}\int_0^{\alpha} a_{j,2S}(\beta)\,d\beta\,\rho_j(\alpha)\,d\alpha. \end{aligned} \tag{2.7$'$}$$

It is known [18, 21] that the ground state of the model is a Dirac sea of $2S$-strings. Thermodynamically this means that in the ground state the density $\rho_{2S}(\lambda)$ is finite, $\tilde{\rho}_{2S}(\lambda)=0$ and $\rho_j(\lambda)=0$ for any $j\neq 2S$. Therefore densities $\tilde{\rho}_{2S}(\lambda)$ and $\rho_j(\lambda)$ have to be considered as the densities of excitations. Using system (2.7) one can express $\rho_{2S}(\lambda)$ in terms of $\tilde{\rho}_{2S}(\lambda)$ and $\rho_j(\lambda)$ with $j\neq 2S$. Substituting this expression into (2.7′) we can find the energy and momentum of the thermodynamic states as a function of densities of excitations over the ground state. We obtain the following result

$$\begin{aligned} E(\rho)&=-N\varepsilon_0+N\int_{-\infty}^{+\infty} s(\alpha)\tilde{\rho}_{2S}(\alpha)\,d\alpha \\ K(\rho)&=N\int_{-\infty}^{+\infty} p(\alpha)\tilde{\rho}_2S(\alpha)\,d\alpha \\ \varepsilon_0&=-\int_{-\infty}^{+\infty} a_{2S,2S}(\alpha)s(\alpha)\,d\alpha \end{aligned} \tag{2.8}$$

where $s(\alpha)$ is defined in the appendix. This formula shows that j-strings with $j\neq 2S$ have zero energy and momentum. The hole in $2S$-strings has the dispersion

$$\varepsilon(\alpha)=s(\alpha)\qquad p(\alpha)=\int_0^{\alpha} s(\beta)\,d\beta-\frac{\pi}{2}\operatorname{sgn}(\alpha) \tag{2.8$''$}$$

† 'Almost' means that the number of other solutions (which exist as shown for example in [20]) is sufficiently small and does not give a contribution to the leading asymptotics of distributions of observables in the thermodynamical limit.

where ε and p are the energy and momentum, respectively. Therefore they describe the degeneration of state.

As a first step in evaluation of scattering amplitudes let us transform system (2.7) in the following way:

$$\begin{aligned}
&s = B_{2S} * \tilde{\rho}_{2S} + \rho_{2S} + \sum_{m=1}^{2S-1} a_m^{2S} * \rho_m + \sum_{n \geq 2S+1} a_{n-2S,1} * \rho_N \\
&\alpha_n * \tilde{\rho}_{2S} = \tilde{\rho}_n + \sum_{m=1}^{2S-1} A_{nm}^{2S} * \rho_m \qquad n < 2S \\
&a_{n-2S,1} * \tilde{\rho}_{2S} = \tilde{\rho}_{n-2S} + \sum_{m \geq 1} A_{n-2S,m-2S} * \rho_m \qquad n > 2S
\end{aligned} \tag{2.8'}$$

where functions $a_{n(\lambda)}^{(r)}$, $A_{nm}^{(r)}(\lambda)$ are given in the appendix. In this system we separate macroscopic densities ρ_{2S}, $\tilde{\rho}_n$, $n \neq 2S$ over the ground state from microscopic ones.

We will search now for the scattering amplitudes in the model using thermodynamic arguments (see, for example, [8]). The main idea is the following. Let us suppose that we have the system in sufficiently large but finite volume. From this we can derive the thermodynamic behaviour of the system from the finite volume quantization of momentums of physical excitations. Let k_j be momentums of these excitations. In the finite volume N of our magnetic chain (N is the number of atoms) we will have the following equations for k_j [22, 23]:

$$\exp(\mathrm{i}k_jN)\xi = S_{jj+1}(k_j, k_{j+1}) \ldots S_{jN}(k_j, k_1)S_{j1}(k_j, k_1) \ldots S_{jj-1}(k_j, k_{j-1})\xi \tag{2.9}$$

where $S_{jl}(k_j, k_l)$ is the amplitude of the scattering of the magnon k_j on the magnon k_l, and ξ is the scattering state.

To find the system of equations for momentum k_j we have to diagonalize the matrices in the RHS of (2.9). Because our system is integrable, the amplitudes $S_{ij}(k, k')$ satisfy the Yang–Baxter equation and the problem of diagonalization of matrices on the RHS is equivalent to the problem of diagonalization of the following transfer matrix [24]

$$t(k) = \mathrm{tr}_0\, S_{01}(k, k_1) \ldots S_{0N}(k, k_N).$$

Therefore we have to recognize the amplitudes $S(k, k')$ from the system (2.8) in order to find the matrix $S(k, k')$ to give us this system from (2.9) with the limits $N \to \infty$, $m \to \infty$ and $m/N = \text{fixed}$.

The following result was obtained in [18] for two magnon states: there are four two-magnon states. This four-dimensional space is a tensor product of two representations of SU(2) of the spin-$\frac{1}{2}$ (each representation describes isotopical states of magnons). The amplitude has the form

$$S(\lambda) = S_t(\lambda; S) \cdot S^{(1/2,1/2)}(\lambda) \cdot \frac{\sinh\left(\frac{\pi}{4S}(\lambda - \mathrm{i})\right)}{\sinh\left(\frac{\pi}{4S}(\lambda + \mathrm{i})\right)} \tag{2.10}$$

Here $S^{(1/2,1/2)}$ is the factorized unitary, crossing-symmetrical S-matrix for particles with spin 1/2 (it is the matrix in $\mathbb{C}^2 \otimes \mathbb{C}^2$)

$$S^{(1/2,1/2)}(\lambda) = \frac{\Gamma(-\mathrm{i}\lambda/2)\Gamma((1+\mathrm{i}\lambda)/2)}{\Gamma(+\mathrm{i}\lambda/2)\Gamma((1-\mathrm{i}\lambda)/2)} \frac{\lambda - \mathrm{i}P}{\lambda - \mathrm{i}} \tag{2.11}$$

where P is the permutation matrix; $P(x\otimes y)=y\otimes x$, and

$$S_t(\lambda;S)=\exp\left(-\mathrm{i}\int_0^\infty \frac{\sin\lambda x}{x}\,\frac{\sinh((S-\frac{1}{2})x)}{\cosh(x/2)\sinh Sx}\,\mathrm{d}x\right).$$

It is easy to see that the multiplier $S_t(\lambda;S)\sinh(\pi/4S(\lambda-\mathrm{i}))/\sinh(\pi/4S(\lambda+\mathrm{i}))$ coincides with one of the eigenvalues of the two-particle ampliutde for the sine-Gordon model. Therefore the simplest conjecture about the structure of the S-matrix in the model (2.1) is

$$S(\lambda)=S^{\mathrm{SG}}(\lambda;S)\otimes S^{(1/2,1/2)}(\lambda) \tag{2.12}$$

where

$$\begin{aligned}S^{\mathrm{SG}}(\lambda;S)=\frac{S_t(\lambda;S)}{\sinh\left(\frac{\pi}{2S}(\lambda-\mathrm{i})\right)}&\left(\left(\sinh\left(\frac{\pi}{2S}(\lambda-\mathrm{i})\right)+\sinh\left(\frac{\pi}{2S}\lambda\right)\right)1\otimes 1\right.\\&+\left(\sinh\left(\frac{\pi}{2S}(\lambda-\mathrm{i})\right)-\sinh\left(\frac{\pi}{2S}(\lambda)\right)\sigma^z\otimes\sigma^z-\sinh\left(\frac{\pi\mathrm{i}}{2S}\right)\right.\\&\left.\times(\sigma^+\otimes\sigma^-+\sigma^-\otimes\sigma^+)\right)\end{aligned}$$

and σ^z, $\sigma^\pm$ are Pauli matrices.

This S-matrix corresponds to the following structure of the n-particle space

$$\mathscr{H}_n\simeq(\mathbb{C}^2)^{\otimes n}\otimes(\mathbb{C}^2)^{\otimes n}. \tag{2.13}$$

But this interpretation is not correct. It is easy to show [15] that the dimension of $\mathscr{H}_n$ are less than $4^n:2^n<\dim\mathscr{H}_n<4^n$ and therefore (2.13) cannot be isomorphic to n-particle space.

Let us find the system of integral equations for thermodynamical densities corresponding to the conjecture (2.12). From equation (2.9) together with the dispersion (2.8″) in the thermodynamic limit, we arrive at the following system:

$$\begin{aligned}&s=B_{2S}*\rho+\tilde\rho+\sum_{m=1}^{2S-1}a_m^{(2S)}*\sigma_m-a_{\bar 1}^{(2S)}*\sigma_{\bar 1}+\sum_{m\geqslant 1}a_m*\rho_m\\&a_m^{(2S)}*\rho=\tilde\sigma_m+\sum_{k=1}^{2S-1}A_{km}^{(2S)}*\sigma_m-s*A_{k,2S-2}^{(2S)}*\sigma_{\bar 1}\\&a_{2S-1}^{(2S)}*\rho=\tilde\sigma_{\bar 1}+\sigma_{\bar 1}-a_2*\sigma_{\bar 1}+\sum_{m=1}^{2S-1}s*A_{2S-2,m}^{(2S)}*\sigma_m\\&a_m*\rho=\tilde\rho_m+\sum_{m\geqslant 1}A_{mn}*\rho_n\end{aligned} \tag{2.14}$$

where the functions a_m, $a_m^{(2S)}$, A_{nm} and $A_{nm}^{(2S)}$ are the same as in (2.8′). Here ρ is the density of the magnons, and $\tilde\rho$ is the density of holes in the distribution of magnons [25]. The densities ρ_n, $\tilde\rho_n$ and σ_n, $\tilde\sigma_n$ respectively appear in the diagonalization of the product of left and right factors of the two-particle S-matrix (see, for example, [8, 22, 23]).

Obviously this system will coincide with (2.8) if we identify $\rho,\tilde\rho,\tilde\sigma_n,\sigma_n$, $n=1,\ldots,2S-1$, $\tilde\rho_n$, ρ_n, $n\geqslant 1$ with $\tilde\rho_{2S}$, ρ_{2S}, $\tilde\rho_n$, ρ_n, $n=1,\ldots,2S-1$, $\tilde\rho_{n+2S}$, ρ_n+2S, $n\geqslant 1$, respectively, and if we suppose that $\tilde\sigma_{\bar 1}=\sigma_1=0$ (in this case there is no equation for σ_1 in (2.14)). This means the space of physical excitations of model (2.1) is the subspace of (2.13). If we make such a conjecture we have to describe the restriction $\sigma_{\bar{1}}\equiv 0$ on the microscopic level, and at that point we will experience major problems. It is an unknown microscopic restriction which gives the states with $\sigma_{\bar{1}}\equiv 0$ in the thermodynamic limits.

Now it is time to remember that we know the restriction which is not exactly the same, but very similar. In [26] it was shown that the spectrum of the transfer matrix of the critical RSOS model can be obtained from the spectrum of the transfer matrix of the 6-vertex model if we put

$$\sigma_{2S-1}=\tilde{\sigma}_{2S-1}=\sigma_{2S-2}=\sigma_{\bar{1}}=\sigma_{\bar{1}}=0$$

in the thermodynamic limit.

Let us use this fact for the interpretation of the integral equations (2.8).

Conjecture. (1) The n-particle space in the model (2.1) over physical vacuum is isomorphic to the tensor product

$$\mathcal{H}_n\simeq\mathcal{H}_n^{\mathrm{RSOS}}\otimes(\mathbb{C}^2)^{\otimes n} \tag{2.15}$$

where $\mathcal{H}_n^{\mathrm{RSOS}}$ is the space of sequences $(a_0, a_1, a_2, \ldots, a_{n-1})$ with $a_{i-1}\perp\frac{1}{2}$, $0\leqslant a_i\leqslant S$, $a_n\equiv a_0\equiv 0$.

(2) If $E(a_0, a_1, \ldots, a_{n-1})$ is the basis in $\mathcal{H}_n^{\mathrm{RSOS}}$ the action of two-particle amplitude on the basis

$$E(a_0, a_1, \ldots, a_{n-1})\otimes e_{i_1}\otimes\ldots\otimes e_{i_n}$$

in (2.15) has the form

$$\begin{aligned}
&S_{jj+1}(\lambda)(E(a_0a_1\ldots a_{n-1})\otimes e_{k_1}\otimes\ldots\otimes e_{k_n})\\
&\quad=\sum_{a_j'}\frac{\Gamma(-\mathrm{i}\lambda/2)\Gamma((1+\mathrm{i}\lambda/2)}{\Gamma(\mathrm{i}\lambda/2)\Gamma((1-\mathrm{i}\lambda)/2)}\begin{bmatrix} & a_j & \\ a_{j-1} & & a_{j+1} \\ & a_j' & \end{bmatrix}(\lambda)\\
&\quad\times E(a_0,\ldots,a_{j-1},a_j',a_{j+1},\ldots,a_{n-1})\otimes e_{k_1}\otimes\ldots\otimes e_{k_{j-1}}\\
&\quad\otimes\left(\frac{\lambda}{\lambda-\mathrm{i}}\,e_{k_j}\otimes e_{k_{j+1}}-\frac{\mathrm{i}}{\lambda-\mathrm{i}}\,e_{k_{j+1}}\otimes e_{k_j}\right)\otimes e_{k_{j+2}}\otimes\ldots\otimes e_{k_n}
\end{aligned} \tag{2.16}$$

where non-zero weights $\begin{bmatrix} & b & \\ a & & c \\ & d & \end{bmatrix}(\lambda)$ are

$$\begin{bmatrix} & l+\frac{1}{2} & \\ l & & l+1 \\ & l+\frac{1}{2} & \end{bmatrix}(\lambda)=\begin{bmatrix} & l-\frac{1}{2} & \\ l & & l-1 \\ & l-\frac{1}{2} & \end{bmatrix}(\lambda)=S_l(\lambda, S+1)$$

$$\begin{bmatrix} & l+\frac{1}{2} & \\ l & & l \\ & l+\frac{1}{2} & \end{bmatrix}(\lambda)=\frac{\sinh\left(\frac{\pi}{2S+2}(\lambda+(2l+1)i)\right)\sin\left(\frac{\pi}{2S+2}\right)}{\sin\left(\frac{\pi}{2S+2}(2l+1)\right)\sinh\left(\frac{\pi}{2S+2}(\lambda-\mathrm{i})\right)}S_l(\lambda, S+1)$$

$$\begin{bmatrix} & l-\frac{1}{2} & \\ l & & l \\ & l-\frac{1}{2} & \end{bmatrix}(\lambda)=\frac{\sinh\left(\frac{\pi}{2S+2}(\lambda-(2l+1)i)\right)\sin\left(\frac{\pi}{2S+2}\right)}{\sin\left(\frac{\pi}{2S+2}(2l+1)\right)\sinh\left(\frac{\pi}{2S+2}(\lambda-\mathrm{i})\right)}S_l(\lambda,S+1)$$

$$\begin{bmatrix} & l-\frac{1}{2} & \\ l & & l \\ & l+\frac{1}{2} & \end{bmatrix}(\lambda)=\begin{bmatrix} & l+\frac{1}{2} & \\ l & & l \\ & l-\frac{1}{2} & \end{bmatrix}(\lambda)$$

$$=\frac{\sinh\left(\frac{\pi\lambda}{2S+2}\right)\left(\sin\left(\frac{2\pi l\lambda}{2S+2}\right)\sin\left(\frac{\pi}{2S+2}(2l+2)\right)\right)^{1/2}}{\sin\left(\frac{\pi}{2S+2}(2l+1)\right)\sinh\left(\frac{\pi}{2S+2}(\lambda-i)\right)}S_l(\lambda,S+1). \tag{2.17}$$

Almost proof of the conjecture. Let us find the integral equations which follow from (2.9) in the thermodynamical limit. Conjecture will be almost proven if these equations will coincide with (2.8).

Using the result of [26] one can show that equation (2.9) gives the following system of linear integral equations if $S(k,k)$ is the matrix (2.16).

$$a_m^{(2S+2)}*\rho=\tilde{\sigma}_m+\sum_{k=1}^{2S}A_{mk}^{(2S+2)}*\sigma_k$$

$$s=\tilde{\rho}+\rho-B*\rho-B^{(2S+2)}*\rho+\sum_{m=1}^{2S}a_m^{(2S-2)}*\sigma_m+\sum_{k\geqslant 1}a_m*\rho_m \tag{2.18}$$

$$a_m*\rho=\tilde{\rho}_m+\sum_{k\geqslant 1}A_{mk}*\rho_k.$$

Here ρ is the density of magnons and $\tilde{\rho}$ is the density of holes in the distribution of magnons. Densities $\tilde{\rho}_n$, ρ_n and $\tilde{\sigma}_n$, σ_n describe the diagonalization of the product of matrices (2.16) in the RHS of (2.9). The functions B and $B^{(2S)}$ are give in the appendix and, as follows from [26], $\tilde{\sigma}_{2S}\equiv 0$.

As in the RSOS model we can exclude the density σ_{2S} from the system (2.18). After this and after the identification ρ, $\tilde{\rho}$, $\tilde{\sigma}_m$, σ_m, $m=1,\ldots,2S-1$, $\tilde{\rho}_n$, ρ_n, $n\geqslant 1$ in (2.18) with $\tilde{\rho}_{2S}$, ρ_{2S}, $\tilde{\rho}_m$, ρ_m, $m=1,\ldots,2S-1$, $\tilde{\rho}_n$, ρ_n, $n\geqslant 1$ in (2.8) respectively, we obtain the system (2.8) from (2.18).

Therefore we can conclude that (2.16) is the most reasonable answer for the S-matrix in the integrable Heisenberg model of the spin S.

Let us make a few remarks about the consequences of this fact.

Remark 1. In the case $S=1$ there is a hidden supersymmetry of the model. It follows from the fact that in this case RSOS part of Hilbert space will be isomorphic to the space of states in the critical Ising model and the action of the supersymmetry will coincide as described by Zamelodchikov in [27]. The value of the central charge in this situation perfectly reflects the fermionic structure of the model

$$c=\tfrac{3}{2}=\tfrac{1}{2}+1.$$

Remark 2. The relation between the value of the central charge in the RSOS model and in the XXX model of the spin S is now clear. The first one is equal to $2-6/r$ (where r is the restriction parameter in the RSOS model), the last one is equal to the $3S/(S+1)$. If we put $r=2S+2$ we will have

$$\frac{3(r-2)}{r}=2-\frac{6}{r}+1.$$

This formula reflects the contribution to the value of the central charge of the factors in the Hilbert space: $2-(6/r)$ from the RSOS factor and 1 from the spin-$\frac{1}{2}$ factor in (2.16).

Remark 3. The dimension of the n-magnon space (2.15) can be evaluated in the following form:

$$\dim \mathscr{H}^{(h)}=\sum_{i_1,\dots,i_{n-1}=1}^{2S+1} K_{1i_1}K_{i_1i_2}\dots K_{i_{n-2}1}\cdot 2^n \tag{2.19}$$

where K is the matrix $(2S+1)\times(2S+1)$:

$$K=\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ & \dots & & 0 & 1 \\ 0 & \dots & & 1 & 0 \end{pmatrix}.$$

Let $\psi(\alpha)$, $\alpha=1,\dots,2S+1$, be the eigenvectors of K:

$$K\psi(\alpha)=2\cos\left(\frac{\pi\alpha}{2S+2}\right)\psi(\alpha)$$

$$\psi_n(\alpha)=\frac{\sin(\pi\alpha n/(2S+2))}{\sqrt{S+1}}.$$

Substituting in (2.19), the spectral decomposition of the matrix K, we have:

$$\dim \mathscr{H}^{(n)}=\sum_{\alpha=1}^{2S+1}\frac{\sin^2(\pi\alpha/(2S+2))}{S+1}\cdot\left(2\cos\left(\frac{\pi\alpha}{2S+2}\right)\right)^n\cdot 2^n.$$

(a) In the limit $n\to\infty$ we have

$$\log(\dim \mathscr{H}^{(n)})=n\log\left(4\cos\left(\frac{2\pi}{2S+2}\right)\right)+\mathrm{O}(1)$$

which agrees with the thermodynamical computations of low-temperature behaviour of the entropy [26].

(b) If n is fixed, and S is sufficeintly large

$$\dim \mathscr{H}^{(n)}=\dim(W_0^{(n)})\cdot 2^n$$

where $\dim W_0^{(n)}$ is the multiplicity of one dimensional subrepresentation in the tensor product of n representations of SU(2) of the spin-$\frac{1}{2}$ [20].

(*c*) If $S = 1$, we have:

$$\dim \mathcal{H}^{(n)} = 2^{n/2-1} \cdot 2^n$$

this is an integer because n is always even. The formula (2.15) agrees with result given in [12].

3. Conclusion

The arguments in section 2 can also be used for derivation of the S-matrix in integrable models of magnets related to any simple Lie algebra. Some of these models are described in [7] for classical Lie algebras. Other examples can be found in [9].

Let me announce here the following result. Suppose that the ground state of the model related to simple Lie algebra $\mathcal{G}$ is the Dirac sea, and is formed by p strings on each level of the Bethe-ansatz equations. It is well known that the physical excitations in this case will be holes in the Dirac sea. The number of excitations will be equal to the rank of the algebra.

Analysis of integral equations for densities in the thermodynamic limit of the Bethe equations gives the following structure of the S-matrix for simple-laced Lie algebras.

The space of n-particle space is the tensor product of two factors. The first one is the space of states of the IRF model [32] related to the algebra $\mathcal{G}$ with the restriction parameter $r = p + h$ where h is the dual Coxeter number. The second factor is isomorphic to $V^{\omega_{j_1}} \otimes \ldots \otimes V^{\omega_{j_n}}$ where j_α are types of excitations on the state, ω_j is the jth fundamental weight of the Lie algebra $\mathcal{G}$ and V^{ω_j} is the irreducible representation of $\mathcal{G}$ with highest weight ω_j.

The S-matrix of the model with respect to factorization of the Hilbert space of the model is the tensor product of two factors. The first one gives the IRF factor [32] in the two-particle amplitude, and thhe second factor of the S-matrix gives the usual chiral Gross–Neveu factor in the S-matrix.

The same structure of the S-matrix has trigonometric generalizations of $\mathcal{G}$-invariant models. The simplest example of these models is the XXZ model of the spin S [5, 28]. Let Δ be the parameter of anisotropy. If $\Delta < 1$ or if $\Delta = \cos\gamma$ and $S < \pi/\gamma$ $(0 < \gamma < \pi/2)$, the S-matrix has the same structure as (2.16), but the second factor becomes the two-particle amplitude for the SG model. The details will be published separately.

One can show that the inhomogeneous *xxx* model of spin S can be considered as a lattice regularization of the WZW model by the operator with the conformal dimension (2.1) [30]. Since scattering amplitudes in the homogeneous XXX model (2.1) coincides with the amplitudes in the inhomogeneous model, the matrix (2.6) should coincide with two-particle amplitude in the perturbed WZW model.

The same answer for the S-matrix in the perturbed WZW model was found by Ahn *et al* [29] from phenomenologic arguments. In [29] the authors propose S-matrices for perturbed coset models. These results can be checked also by comparing with Bethe-ansatz computations for RSOS-models [31].

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Appendix

We use the following normalization of Fourier transform:

$$a(\lambda) = \int_{-\infty}^{+\infty} e^{i\lambda x}\, \hat{a}(x)\, \frac{dx}{2\pi} \qquad \hat{a}(x) = \int_{-\infty}^{+\infty} e^{-i\lambda x}\, a(\lambda)\, d\lambda.$$

Fourier images of functions mentioned in the main text are:

$$\hat{a}_{nm}(x) = \hat{s}(x)\hat{A}_{nm}(x)$$

$$\hat{A}_{mn}(x) = \hat{A}_{mn}(x) = 2e^{-n|x|/2} \coth\left(\frac{x}{2}\right) \sinh\left(\frac{mx}{2}\right) \qquad n \geqslant m$$

$$\hat{a}^{(r)}_{nm}(x) = \hat{s}(x)\hat{A}^{(r)}_{nm}(x)$$

$$\hat{A}^{(r)}_{nm}(x) = \hat{A}^{(r)}_{mn}(x) = 2\,\frac{\sin\left(\frac{(r-n)x}{2}\right)}{\sinh\left(\frac{rx}{2}\right)} \coth\left(\frac{x}{2}\right) \sinh\left(\frac{mx}{2}\right) \qquad n \geqslant m$$

$$\hat{B}_n(x) = \frac{1}{\hat{A}_{n,n}(x)}$$

$$\hat{s}(x) = \frac{1}{2\cosh\left(\frac{x}{2}\right)}.$$

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